Markowitz Portfolio Construction Using CVXPY

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[1]With Stephen Boyd, Ronald Kahn, Philipp Schiele, and Thomas Schmelzer [Boyd et al., 2024].

Markowitz portfolio construction: Challenges & contributions

challenges

- Markowitz portfolio construction balances risk and return through convex optimization
- the basic version can be sensitive to estimation errors, often producing impractical portfolios

contributions

- collect minimal set of constraints and extensions from prior work to address practical issues
  - constraints on leverage, turnover, etc. [Grinold & Kahn, 2000]
  - address uncertainty with robust optimization [Ben-Tal, El Ghaoui, & Nemirovski, 2009]
  - incorporate soft constraints in optimization problems [Bertsimas & Brown, 2011]
- novel method for how to prioritize constraints
- extension preserves convexity: **easily implemented in CVXPY**
- extensive empirical evaluation on historical data

Basic Markowitz optimization

$$\begin{array}{ll} \text{maximize} & \mu^T w \\ \text{subject to} & w^T \Sigma w \leq (\sigma^{\text{tar}})^2, \quad \mathbf{1}^T w = 1 \end{array}$$

- variable $w \in \mathbf{R}^n$ of portfolio weights
- $\mu \in \mathbf{R}^n$ and $\Sigma \in \mathbf{S}^n_{++}$ are asset return mean and covariance
- σ^{tar} is target (per period) volatility
- basic form goes back to [Markowitz, 1952]

```
w = cp.Variable(n)
objective = mu.T @ w
constraints = [cp.quad_form(w, Sigma) <= sigma**2, cp.sum(w) == 1]
prob = cp.Problem(cp.Maximize(objective), constraints)
prob.solve()
```

Critiques of Markowitz optimization

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- sensitivity to data errors and estimation uncertainty
- risk symmetry
- maximizing expected utility versus mean-variance
- statistical assumptions: assumes Gaussian returns, and ignores higher moments
- greedy method, only looks one step ahead

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- we address the first issue of sensitivity to data errors and estimation uncertainty
- the other critiques seem less relevant in practice [Luxenberg and Boyd, 2023]

Adding practical constraints and objective terms

- include cash holdings c, previous holdings w^{pre}, trades $z = w - w^{\text{pre}}$
- account for (convex) holding costs ϕ^{hold} and trading costs ϕ^{trade}
- limit weights, cash, trades, turnover $T = \|z\|_1$, and leverage $L = \|w\|_1$

$$
\begin{array}{ll}
\text{maximize} & \mu^T w - \gamma^{\text{hold}} \phi^{\text{hold}}(w, c) - \gamma^{\text{trade}} \phi^{\text{trade}}(z) \\
\text{subject to} & \mathbf{1}^T w + c = 1, \quad z = w - w^{\text{pre}}, \\
& w^{\text{min}} \leq w \leq w^{\text{max}}, \quad c^{\text{min}} \leq c \leq c^{\text{max}}, \quad L \leq L^{\text{tar}}, \\
& z^{\text{min}} \leq z \leq z^{\text{max}}, \quad T \leq T^{\text{tar}}, \\
& \|\Sigma^{1/2} w\|_2 \leq \sigma^{\text{tar}}
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remaining challenges (and solutions)

- Σ is estimated in factor covariance form; estimating μ is difficult and typically proprietary
- optimization is sensitive to errors in μ and Σ (use robustification)
- constraints may lead to infeasibility or unnecessary trading (use soft constraints)

Factor covariance model

$$\Sigma_t = F_t \Sigma_t^{\mathrm{f}} F_t^T + D_t$$

- $F_t \in \mathbf{R}^{n \times k}$ is matrix of factor loadings
- k is number of factors, typically with $k \ll n$
- Σ_t^{f} is $k \times k$ factor covariance matrix
- D_t is diagonal matrix of unexplained (idiosyncratic) variances
- a strong regularizer which can give better return covariance estimates
- factors constructed by many methods, like principal component analysis (PCA) or by hand

Computational benefits of factor model

- with factor model, cost of portfolio optimization reduced from $O(n^3)$ to $O(nk^2)$ flops [Boyd and Vandenberghe, 2004]
- easily exploited in modeling languages like CVXPY
- timings for Clarabel open source solver:

		solve time (s)	
assets n	factors k	factor model	full covariance
100	10	0.002	0.040
300	20	0.010	0.700
1000	30	0.080	25.600
3000	50	0.600	460.000

Robustifying Markowitz

- basic Markowitz optimization can be sensitive to estimation errors in μ, Σ
- replace mean return $\mu^T w$ with worst-case return

$$R^{\text{wc}} = \min\{(\mu+\delta)^T w \mid |\delta| \leq \rho\} = \mu^T w - \rho^T |w|$$

  where $\rho \geq 0$ is vector of mean return uncertainties
- replace risk $w^T \Sigma w$ with worst-case risk

$$\begin{aligned}(\sigma^{\text{wc}})^2 &= \max\{w^T(\Sigma+\Delta)w \mid |\Delta_{ij}| \leq \varrho(\Sigma_{ii}\Sigma_{jj})^{1/2}\} \\ &= \sigma^2 + \varrho\left(\sum_{i=1}^n \Sigma_{ii}^{1/2}|w_i|\right)^2\end{aligned}$$

  where $\varrho \geq 0$ represents covariance uncertainty
- easily handled by CVXPY

Softening constraints

- **soft constraints** allow limited violations of constraints, based on priority
- to soften a constraint $f \leq f^{\text{max}}$, replace it with a penalty term $\gamma(f - f^{\text{max}})_+$ in the objective
- in Markowitz risk, leverage, and turnover can be softened, giving three priority parameters

$$\gamma^{\text{risk}}, \quad \gamma^{\text{lev}}, \quad \gamma^{\text{turn}}$$

- the softened problem reduces unnecessary trading and is always feasible

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choosing priority parameters

- can be chosen or initialized based on Lagrange multipliers of hard constrained problem
- *e.g.*, as 80th percentile of recorded multipliers over a historical period
- fast solve time enables backtesting to fine-tune parameters

Implementation in CVXPY

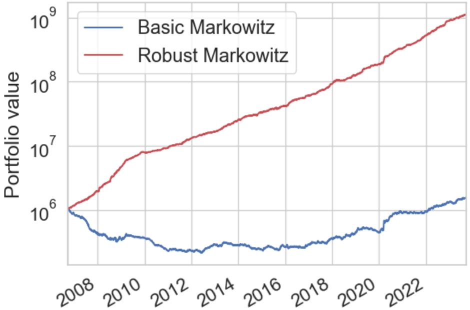
```
w, c, z = cp.Variable(n_assets), cp.Variable(), cp.Variable(n_assets)

return_wc = mu @ w - rho_mean @ cp.abs(w)
risk_uncertainty = rho_covariance ** 0.5 * volas @ cp.abs(w)
risk_wc = cp.norm2(cp.hstack([cp.norm2(chol.T @ w), risk_uncertainty]))

objective = (
    return_wc
    - param.gamma_hold * kappa_short @ cp.pos(-w)
    - param.gamma_trade * kappa_spread @ cp.abs(z)
)

constraints = [cp.sum(w) + c == 1,
    w_min <= w, w <= w_max, c_min <= c, c <= c_max,
    z_min <= z, z <= z_max, z == w - w_prev,
    cp.norm1(z) <= T_tar, cp.norm1(w) <= L_tar, risk_wc <= sigma_tar]

cp.Problem(cp.Maximize(objective), constraints).solve()
```

Data and experimental setup

- S&P 100 stocks, data gathered daily from 2000-01-04 to 2023-09-22
- exclude stocks without data for the full period (gives $n = 74$ assets)
- simulated but realistic mean predictions, and EWMA covariance
- priority parameters retuned each year based on the previous two years of data

- **focus**: relative performance comparison of methods, not real portfolio construction

Portfolio performance



Metric	Basic	Robust
Return	3.5%	38.1%
Risk	14.4%	8.6%
Sharpe	0.2	4.6
Drawdown	80%	6%

- out-of-sample portfolio performance for basic Markowitz and robust Markowitz

Conclusions

- basic Markowitz optimization can be sensitive to estimation errors and uncertainties
- extended to include practical constraints (*e.g.*, leverage, turnover, . . .)
- addressed estimation errors with robust optimization
- leveraged soft constraints to reduce trading and ensure feasibility
- can be handled nicely with modern domain-specific languages like CVXPY

Thank you!